Comparative Static Analysis: Partial Differentiation Relationships in Equilibrium

Dylan Bollard

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1 Introduction:

The class so far has covered components of linear algebra, uni-variate calculus and corresponding applications of these mathematical tools for economic analysis. The most interesting application so far that I've found is comparative static analysis. A system that allows us to find how small changes in things like tax rates and government spending can affect desired national income or consumption is not only extremely useful, but has a wide array of applications outside of a national income model. For the purposes of this paper, we apply this system to a national income model, given that macroeconomic trends will be obvious and are supported well with available theory.

Throughout this assessment, macro- (and some micro-) economic theory is used to clarify findings and make the connection between it and the mathematical model more apparent. In addition, historical context is considered when applicable. The main mathematical tools to be applied are partial differentiation, to show how changes in one of these controllable macroeconomic parameters can affect our desired income, consumption or tax revenue, and a brief use of matrix inversion as a proof of concept.

2 Models:

The following national-income model will be used as the stepping stone into deeper analysis:[1]

$$Y = C + I_0 + G_0 \tag{2.1}$$
$$C = \alpha + \beta(Y - T) \qquad (\alpha > 0; \quad 0 < \beta < 1) \tag{2.2}$$
$$T = \gamma + \delta Y \qquad (\gamma > 0; \quad 0 < \delta < 1) \tag{2.3}$$

The primary goal should be to find an equilibrium using these equations. Equilibrium will give us a broader sense of how we define our primary variables (outcomes for income, consumption and tax revenue) in terms of all of our controllable parameters and larger macroeconomic trends. With these equations there are three groups of variables. The sum of all the income received for contributing resources to GDP is called national income (Y). At some points in the discussion that follows, it will be useful to refer to real GDP as national income.

1. Endogenous variables:
   - Y (national income)
   - C (consumption)
   - T (taxes)

[1] Alpha C. Chiang and Kevin Wainwright, Fundamental Methods of Mathematical Economics (Boston, MA: McGraw-Hill/Irwin, 2005), 172-175.

2. Exogenous variables:
   - I_0 (investment)
   - G_0 (government expenditure)
3. Parameters:
   - α (base consumption)
   - β (marginal propensity to consume)
   - δ (income tax rate)
   - γ (positive government tax revenue)

The equations presented represent national-income (Y), consumption (C) and taxes (T). For example, national consumption is a function of income and taxes as well, so it is considered an endogenous variable because it is dependent on other functions. In fact, Y, C, and T are endogenous (dependent) given their relationship to one another.

I_0 (investment) and G_0 (government expenditure) are part of each equation, and changes in them will cause changes in equilibrium. I_0 and G_0 are classified as exogenous (independent) variables because even though they affect our model, they themselves are not affected by that model. These variables exist for setting arbitrary external conditions. More realistically though, even though the model wont affect the value of investment or government expenditure, they become useful policy tools in addressing desired changes in equilibrium of our endogenous variables. For example, as we'll see, an increase in consumption could be achieved by an increase in government spending.

In addition, our third category of variables are labeled as parameters given that they have definite values and exist to help us better understand how an equilibrium is established. α (base consumption) must be positive because even if $Y - T = 0$, there is still consumption in the economy. β (marginal propensity to consume) is a measurement of one's willingness to purchase goods and services, it's strictly positive, and is less than 1. δ (income tax rate) is positive for obvious reasons. Finally, γ (tax revenue) is strictly positive given that even if income were 0, the government would still collect some manner of taxes.

By first finding equilibrium within the system in terms of exogenous variables and parameters, the basis for partial differentiation will be readily available. Each partial derivative will show the relationships between endogenous variables and the effect of various parameters on them.

3 Methods:

In order to find equilibrium, we solve the system of equations in terms of their exogenous variables.

$$
\begin{aligned}
Y &= C + I_0 + G_0 \\
C &= \alpha + \beta(Y - T) && (\alpha > 0; \quad 0 < \beta < 1) \\
T &= \gamma + \delta Y && (\gamma > 0; \quad 0 < \delta < 1)
\end{aligned}
$$

Can be rearranged as:

$$Y - C = I_0 + G_0 \tag{3.1}$$
$$C + \beta T - \beta Y = \alpha \qquad (\alpha > 0; \quad 0 < \beta < 1) \tag{3.2}$$
$$T - \delta Y = \gamma \qquad (\gamma > 0; \quad 0 < \delta < 1) \tag{3.3}$$

At this point we can either rearrange the equations algebraically in terms of just parameters or we can use a matrix to solve for Y, C, and T.

$$
\underset{\text{A}}{\begin{bmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{bmatrix}}
\underset{\mathbf{x}}{\begin{bmatrix} Y \\ C \\ T \end{bmatrix}}
\underset{=}{=}
\underset{\mathbf{b}}{\begin{bmatrix} I_0 + G_0 \\ \alpha \\ \gamma \end{bmatrix}}
$$

Given that,

$$
\begin{aligned}
A\mathbf{x} &= \mathbf{b} \\
A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\
\mathbf{x} &= A^{-1}\mathbf{b}
\end{aligned}
$$

Solving for A^{-1} will give the equilibrium levels of Y, C, and T.

$$
A^{-1} = \frac{1}{1 - \beta + \delta\beta} \begin{bmatrix} 1 & 1 & -\beta \\ -\delta\beta + \beta & 1 & -\beta \\ \delta & \delta & 1 - \beta \end{bmatrix}
$$

$$
\begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \frac{1}{1 - \beta + \delta\beta} \begin{bmatrix} 1 & 1 & -\beta \\ -\delta\beta + \beta & 1 & -\beta \\ \delta & \delta & 1 - \beta \end{bmatrix} \begin{bmatrix} I_0 + G_0 \\ \alpha \\ \gamma \end{bmatrix}
$$

Finally, evaluating the matrix multiplication will yield the following values:[2]

[2] Alpha C. Chiang and Kevin Wainwright, Fundamental Methods of Mathematical Economics (Boston, MA: McGraw-Hill/Irwin, 2005), 172-175.

$$Y^* = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \delta\beta} \tag{3.4}$$

$$C^* = \frac{\alpha - \beta\gamma + (I_0 + G_0)(\beta - \delta\beta)}{1 - \beta + \delta\beta} \tag{3.5}$$

$$T^* = \frac{\delta(I_0 + G_0 + \alpha) + \gamma(1 - \beta)}{1 - \beta + \delta\beta} \tag{3.6}$$

These equations represent equilibrium levels of all three in the economy with respects to the various parameters from the original equations. At this point we could use real world values and we might be able to discern trends if we spent enough time working the equations. Solving for partial derivatives however will tell us exactly how a change in one of our parameters will effect each equilibrium level.

We start with how changes in our variables will effect equilibrium income. A few things to note first however:[3]

$$\begin{aligned} \Delta C &= \beta \Delta Y && \text{Where consumption is function of } \beta \text{ (\textit{marginal propensity to consume}) and income } Y \\ \frac{\Delta C}{\Delta Y} &= \beta \\ \frac{\partial C}{\partial Y} &= \beta && \text{Note that this is C and Y, not C* and Y*}^4 \end{aligned} \tag{3.7}$$

$$1 - \beta + \beta\delta > 0 \quad \Leftarrow \quad \beta > 0$$

Generally we'd just partially derive with respect to I_0, G_0, δ, γ, because those are macroeconomic values that can be directly manipulated by government policy.

3.1 *Income (Y*)*:

$$\frac{\partial Y^*}{\partial G_0} = \frac{1}{1 - \beta + \beta\delta} \quad > 0 \tag{3.1.1}$$

The first example reads as "the rate in change of equilibrium income with respects to the rate in change of government expenditure" and is called the *government expenditure*

[3] Subho Mukher, "Keynes' Theory of Investment Multiplier (With Diagram)," Economics Discussion, August 26, 2015.

[4] $\frac{\partial y}{\partial x}$, is read as "the change in y with respect to the change in x"

multiplier.[5] [6] This shows the "ripple" effect government expenditure causes in the economy. This multiplier represents the multiple by which GDP increases or decreases in response to an increase or decrease in government expenditures or investment. This multiplier represents an equilibrium, or a trade off rather, of two other important topics. If we take for example country A, with a marginal propensity to consume (β) of 0.8, the multiplier is rather large. This means that an increase in government spending or investment would have a large impact (in the form of a spending multiplier) on not only equilibrium income but also on GDP as a whole. The size of the multiplier is critical and was a key element in recent discussions of the effectiveness of the Obama administrations fiscal stimulus package, officially titled the American Recovery and Reinvestment Act of 2009.[7]

In **(2.3)** we include a flat rate for taxes as well as tax being a function of income. In more general models the latter is sometimes left off and we're left with a basic picture of how this relationship works. [8]

$$Spending\ Multiplier = \frac{1}{1-\beta} \tag{3.1.2}$$

The higher our *marginal propensity to consume*, the more impact an increase in government expenditure will have on a positive increase in equilibrium income. As a result of the multiplier effect, small changes in investment or government spending can create very large changes in total output. A positive aspect of the multiplier effect is that macroeconomic policy can effect substantial improvements with relatively small amounts of autonomous expenditures. The reason for the expenditure multiplier is that one persons spending becomes another persons income, which leads to additional spending and additional income, and so forth, so that the overall impact on equilibrium income is larger than the initial increase in spending.[9]

$$\frac{\partial Y^*}{\partial I_0} = \frac{1}{1-\beta+\beta\delta} \quad > 0 \tag{3.1.3}$$

This example reads as "the rate in change of equilibrium income with respects to the rate in change of investment" and is called the *investment money multiplier*, first introduced by Kahn but later defined by Keynes. An investment multiplier refers to the concept that any increase in public or private investment spending has a largely positive impact on aggregate income. The larger an investment's multiplier, the more efficient it is at creating and distributing wealth throughout an economy.

[5] $\frac{\Delta Y}{\Delta Spending} > 1$

[6] Government expenditure in the United States is about 20 percent of GDP.

[7] The American Recovery and Reinvestment Act was developed during the Great Recession as a ways to stop job losses and attempt to create jobs at the same time. It also offered immediate assistance for those most affected by the recession and invested in infrastructure, education, health, and renewable energy.

[8] Subho Mukher, "Keynes' Theory of Investment Multiplier (With Diagram)," Economics Discussion, August 26, 2015.

[9] Steven A. Greenlaw et al., Principles of Macroeconomics (Houston, TX: Openstax College, Rice University, 2016), 293.

Where as the *Spending Multiplier* is a function of the *marginal propensity to consume*, the *Investment Money Multiplier* is a function of investment and the rate at which capital is generated and added to circulation. For example, if the *Investment Money Multiplier* is high, firm A's initial building of a power plant has a ripple effect on the economy in that the investment generates capital creation and distribution larger than the initial investment. Investment expenditure refers to purchases of physical plant and equipment, primarily by businesses.[10] [11]

$$\frac{\partial Y^*}{\partial \gamma} = \frac{-\beta}{1-\beta+\beta\delta} \quad < 0 \tag{3.1.4}$$

This example reads as "the rate in change of equilibrium income with respects to the rate in change of government tax revenue". This function will always be negative given that an increase in tax revenue means an increase in taxes, *ceteris parabus*. An increase in taxes would surely lower individual income given (**2.3**). This relationship is one of the more debated topics in politics, but here it's boiled down to it's basis. A decrease in tax revenue, in this model, would show an increase in income and by extension an increase in consumption and investment. The reason that it's essentially the spending multiplier scaled by the *marginal propensity to consume*, is a two part relationship. First, it's the negative spending multiplier because we're pulling money out of circulation and income. Second, it's scaled because of (**3.7**), where Δ income and Δ consumption are decreasing. Because these two relationships are negative, algebraically they create a positive value for β, however due to the previous explanation, the net change in income is negative.

$$\frac{\partial Y^*}{\partial \delta} = \frac{-\beta Y*}{1-\beta+\beta\delta} \quad < 0 \tag{3.1.5}$$

Similarly,

$$\Delta C = \beta \Delta Y^{12} \qquad \text{and } \Delta Y \text{ is } \textit{negative} \text{ given } \mathbf{3.1.5}$$
$$\Delta C = -\beta \Delta Y$$

where,

$$\lim_{\Delta Y \to 0} -\beta \Delta Y = -\beta Y^{*13}$$

[10]If Starbucks builds a new store, or Amazon buys robots, these expenditures are counted under business investment. Investment demand is far smaller than consumption demand, typically accounting for only about $15-18$*percent* of GDP, but it is very important for the economy because this is where jobs are created.

[11]Steven A. Greenlaw et al., Principles of Macroeconomics (Houston, TX: Openstax College, Rice University, 2016), 523.

[12]$\frac{\Delta C}{\Delta Y} = \beta$ is the fundamental relationship for marginal propensity to consume.

[13]given that no change in income would mean equilibrium has been established

The next partial derivative is taken to be the effect a change in tax rate has on the equilibrium income. Given that it will always be negative, we can assume an increase in tax would lower equilibrium income. It's not a multiplier, however, but it does explicitly show the relationship. An increase in the tax rate lowers income, as you'd expect it to.[14] The reason for this scaling of $-\beta Y^*$ is somewhat simple. Whereas previously we looked at the rate in change of equilibrium income with respect to the rate in change of tax revenue (γ), and discovered it was scaled by $-\beta$, a change in tax rate is simply the same relationship with one addition. Tax revenue is a function of income and the tax rate. So an increase in the tax rate (δ) will be an increase in tax revenue (γ) given the equilibrium income.

The individual partial derivatives, and by extension their relationship to equilibrium income, were each summarized individually. The total derivative is a way for us to look at the total change in equilibrium income (Y^*) with respect to the sum of the changes to each of the individual variables.

$$x_i = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} G_0 \\ I_0 \\ \gamma \\ \delta \\ \vdots \\ \text{other economic indicators}^{15} \end{bmatrix}$$

$$\begin{aligned} \text{Total Derivative } Y^* = \frac{dY^*}{dx_i} &= \sum_{i=1}^{n} \frac{\partial Y^*}{\partial X_i} \Delta x_i \\ &= \frac{\partial Y^*}{\partial G_0} \Delta G_0 + \frac{\partial Y^*}{\partial I_0} \Delta I_0 + \frac{\partial Y^*}{\partial \gamma} \Delta \gamma + \frac{\partial Y^*}{\partial \delta} \Delta \delta \end{aligned} \qquad (3.1.6)$$

3.2 ***Consumption (C*)***:

$$\frac{\partial C}{\partial (I_0 + G_0)} = \frac{\beta(1-\delta)}{1-\beta+\beta\delta} \quad > 0 \qquad (3.2.1)$$

This is the relationship between government spending and national consumption. Given that government spending <u>is</u> consumption in this model, this value is always positive. A quick note about government spending and GDP first however. When the government spends more than it makes from tax revenue, there exists a *deficit*. It's important to note that positive GDP can be obtained even when the government has a budget deficit, as is usually the

[14] Alpha C. Chiang and Kevin Wainwright, Fundamental Methods of Mathematical Economics (Boston, MA: McGraw-Hill/Irwin, 2005), 172-175.

[15] other economic indicators are left out due to the scope of this exercise. A variety of variables exist including import/export figures and unemployment rates that effect all of these individual relationships. The goal is simply to provide a basic model for discussion.

case. By spending money, the government is putting it into national circulation, and raising consumption, at the rate of the *spending multiplier*.

A similar relationship to government expenditure and consumption exists here between private/public sector investment and consumption. As investment increases, money is being created within the economy by way of the banking system. In this respect, the government is following *expansionary monetary policy*.[16] This type of policy is characteristic of lowered interest rates and an increase in spending and investment.

$$\frac{\partial C}{\partial \gamma} = \frac{-\beta}{1-\beta+\beta\delta} \quad < 0 \tag{3.2.3}$$

In this relationship between government revenue and consumption, the opposite effect is desired. If we wish to raise tax revenue, we take money out of circulation and attempt to lower spending and GDP. As positive tax revenue rises, the rate of consumption declines. This is characteristic of *contractionary monetary policy*. The reason it's scaled negatively by the spending multiplier is obvious, given that we're pulling money out of circulation. As explained earlier following (**3.1.4**), Δ income and Δ consumption are decreasing as a result of the previous finding.

$$\frac{-\Delta C}{-\Delta Y} = \beta$$
$$\frac{\Delta C}{\Delta Y} = \beta$$

$$\frac{\partial C}{\partial \delta} = \frac{-\beta(I_0 + G_0 + C^*)}{1-\beta+\beta\delta} \quad < 0 \tag{3.2.4}$$

Because the change in *consumption* as a result of changes in the tax rate is negative, raising taxes is an example of *fiscal policy*. *Expansionary fiscal policy*, used during recessions as a mean to stimulate the economy, takes the form of tax cuts and increased government spending. *Contractionary fiscal policy* would take the form of increased tax rates or lowering government spending, so as to quell consumption. This might be done when inflation is high. All in all, this relationship between the tax rate and consumption is used when addressing a multitude of economic issues. An increase in the income tax rate will cause a decrease in consumption, *ceteris parabus*. Simply for the fact that less money is in public circulation and that consumption is a function of the tax rate on income as in (**2.2**).

$$\begin{aligned}\text{Total Derivative } C^* = \frac{dC^*}{dx_i} &= \sum_{i=1}^{n} \frac{\partial C^*}{\partial X_i}\Delta x_i \\ &= \frac{\partial C^*}{\partial G_0}\Delta G_0 + \frac{\partial C^*}{\partial I_0}\Delta I_0 + \frac{\partial C^*}{\partial \gamma}\Delta\gamma + \frac{\partial C^*}{\partial \delta}\Delta\delta\end{aligned} \tag{3.2.5}$$

[16]Steven A. Greenlaw et al., Principles of Macroeconomics (Houston, TX: Openstax College, Rice University, 2016).

3.3 ***Taxes (T*)***:

These indicators are perhaps less interesting, but further the notion that we can break down each of these relationships into small pieces.

Keeping in mind that,

$$\frac{\partial Y^*}{\partial(I_0 + G_0)} = \frac{1}{1 - \beta + \beta\delta} \quad > 0$$

is the rate at which government expenditure and private/public investment changes income, it would make sense then that that rate of change in income, multiplied by the tax rate (δ), would be the added tax revenue due to a change in government expenditure or private/public investment.

$$\frac{\partial T}{\partial(I_0 + G_0)} = \frac{\delta}{1 - \beta + \beta\delta} \quad > 0 \tag{3.3.1}$$

There is also a positive relationship between investment and tax revenue. Again, the more wealth that is created, the more tax revenue. Also, the amount of new tax revenue generated by investment is actually just the tax rate multiplied by the money multiplier, the rate at which money is created due to investment. This makes sense because if one was to generate x-real dollars of income, one would owe $\delta * x$ real dollars in extra revenue.

$$\frac{\partial T}{\partial \gamma} = \frac{1 - \beta}{1 - \beta + \beta\delta} \quad > 0 \tag{3.3.3}$$

This relationship represents the change in tax revenue given a change in *desired* tax revenue. This may seem simple but it's an interesting relationship. If the government were to increase *desired* tax revenue, tax revenue would increase. The *marginal propensity to consume* (MPC) + the *marginal propensity to save* (MPS) = 1, because each new unit of income is either saved or spent. As tax revenue goes up, taxes must be increased, and when taxes increase, people have less money to spend. When they have less to spend, the MPC declines,[17] and the MPS to increases. So our multiplier is being scaled by the MPS as tax revenue increases.

$$\begin{aligned} \frac{\partial T}{\partial \delta} &= \frac{-\beta Y^* + Y^*}{1 - \beta + \beta\delta} \\ &= \frac{Y^*(1 - \beta)}{1 - \beta + \beta\delta} \quad > 0 \quad \Leftarrow \beta < 1 \therefore -\beta Y^* < Y^* \end{aligned} \tag{3.3.4}$$

Here, we look at how the tax rate effects tax revenue. This relationship is the same as before, except that when the tax rate rises, it's scaled by the MPS, <u>and</u> is a function

[17] Because MPS + MPC = 1, MPC = 1 - MPS and MPS = 1 - MPC

of income. Obviously, raising the tax rate will increase tax revenue. But at this point, we can deduce whether or not an increase in taxes will be the *most effective* way to increase tax revenue. Because this relationship isn't a multiplier, that is that it doesn't relate a percentage change in tax rates to a dollar amount change in tax revenue, we can't say for sure if it's the best approach. It would be appropriate to remind that this is a basic model and only takes changes in one variable into account.

$$\begin{aligned}\text{Total Derivative } T^* = \frac{dT^*}{dx_i} &= \sum_{i=1}^{n} \frac{\partial T^*}{\partial X_i}\Delta x_i \\ &= \frac{\partial T^*}{\partial G_0}\Delta G_0 + \frac{\partial T^*}{\partial I_0}\Delta I_0 + \frac{\partial T^*}{\partial \gamma}\Delta\gamma + \frac{\partial T^*}{\partial \delta}\Delta\delta \qquad (3.3.5)\end{aligned}$$

Changes in multiple variables are more complicated and will be left for future exercises. All of these individual relationships can be summarized neatly with previous work, given that all these partial derivatives are the rates in change of endogenous variables with respect to changes in exogenous variables and a few parameters.

The reason that the following relationships were not explored should be clear, but hopefully the following contextual summary will show why. α (base consumption) is just an arbitrary value used to show that consumption is never actually 0. Measuring the rates in change of income or consumption with respects to α doesn't really make sense because it itself can't be manipulated. For the following summary of ideas to make sense however, their respective derivatives have been included.

$$\frac{\partial Y}{\partial \alpha} = \frac{1}{1-\beta+\beta\delta}$$
$$\frac{\partial C}{\partial \alpha} = \frac{1}{1-\beta+\beta\delta}$$
$$\frac{\partial T}{\partial \alpha} = \frac{\delta}{1-\beta+\beta\delta}$$

The following is an interesting explanation again why these partial derivatives compose the inverse of our original equations (**2.2-2.4**).

$$\begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x^* \end{bmatrix}$$
$$\begin{bmatrix} I_0 + G_0 \\ \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} d^* \end{bmatrix}$$

$$
\begin{aligned}
x^* &= A^{-1}d^* \\
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \\
&= \begin{bmatrix} a_{11}d_{11} & a_{12}d_{12} & a_{13}d_{13} \\ a_{21}d_{21} & a_{22}d_{22} & a_{23}d_{23} \\ a_{31}d_{31} & a_{32}d_{32} & a_{33}d_{33} \end{bmatrix} \\
\frac{\partial x_i}{\partial d_j} &= a_{ij} \\
\frac{\partial x^*}{\partial d^*} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\end{aligned}
$$

$$
\frac{\partial x^*}{\partial d^*} = \begin{bmatrix} \frac{\partial Y}{\partial I_0 + G_0} & \frac{\partial Y}{\partial \alpha} & \frac{\partial Y}{\partial \gamma} \\ \frac{\partial C}{\partial I_0 + G_0} & \frac{\partial C}{\partial \alpha} & \frac{\partial C}{\partial \gamma} \\ \frac{\partial T}{\partial I_0 + G_0} & \frac{\partial T}{\partial \alpha} & \frac{\partial T}{\partial \gamma} \end{bmatrix} = \frac{1}{1 - \beta + \delta\beta} \begin{bmatrix} 1 & 1 & -\beta \\ -\delta\beta + \beta & 1 & -\beta \\ \delta & \delta & 1 - \beta \end{bmatrix} = A^{-1}
$$

This is an interesting way of beginning to think about setting up all of these equations so that we can see how a change in one variable might affect the equilibrium of all of the endogenous variables. If we wished to increase income, what do we change? Is increasing government spending on infrastructure, or lowering the individual tax rate a better way to increase social utility? Obviously, these questions are far beyond the scope of this paper, and require much larger equations with vastly richer relationships. But this system is the *basis* for further exploration into comparative statics.

4 Application:

Given the following values, what are the values of each relationship? That is, what would be the best application of resources to minimize effort given desired changes in each variable? For that answer we look at the value of each partial derivative and treat them as scalars. The largest scalar will be the one that has the largest impact on the change in the endogenous variable we seek to change. [18]

$$\alpha = 200 \qquad \beta = 0.8$$
$$\gamma = 120 \qquad \delta = 0.25$$
$$I_0 = 300 \qquad G_0 = 200$$

$$\frac{\partial Y^*}{\partial I_0 + G_0} = 2.5 \qquad \frac{\partial Y^*}{\partial \alpha} = 2.5 \qquad \frac{\partial Y^*}{\partial \gamma} = -2 \qquad \frac{\partial Y^*}{\partial \delta} = -2Y^*$$
$$\frac{\partial C^*}{\partial I_0 + G_0} = 1.5 \qquad \frac{\partial C^*}{\partial \alpha} = 2.5 \qquad \frac{\partial C^*}{\partial \gamma} = -2 \qquad \frac{\partial C^*}{\partial \delta} = 1.6(C^* - 625)$$
$$\frac{\partial T^*}{\partial I_0 + G_0} = 0.625 \qquad \frac{\partial T^*}{\partial \alpha} = 2.5 \qquad \frac{\partial T^*}{\partial \gamma} = 0.5 \qquad \frac{\partial T^*}{\partial \delta} = 0.5Y^*$$

$$\text{Total Derivative } Y^* = 2.5\Delta G_0 + 2.5\Delta I_0 + 2.5\Delta\alpha + (-2)\Delta\gamma$$ [19]

$$\text{Total Derivative } C^* = 1.5\Delta G_0 + 1.5\Delta I_0 + 2.5\Delta\alpha + (-2)\Delta\gamma$$

$$\text{Total Derivative } T^* = 0.625\Delta G_0 + 0.625\Delta I_0 + 2.5\Delta\alpha + 0.5\Delta\gamma$$

Given that a change in basic consumption (α) can't be controlled by government policy, the next highest values would effect the overall change the most. To change income, the best approach would be to increase government spending (infrastructure, expand government job opportunities, etc.). To change consumption, the same approach would be the most beneficial, which makes sense given that an increase in consumption precludes an increase in income. In addition, the parameter that increases income the most would also increase tax revenue the most. However, the tax rate would closely do the same work.

5 Conclusion:

This model is meant to be as general as possible, but this venture into the intricate relationships of each variable has an interesting number of takeaways. Positively changing income isn't as simple as higher rates of pay. It isn't as simple as decreasing taxes either. All of these factors exist in an inter-related system known as the economy. Before this class

[18] "The Keynesian Model Of Income Determination In A Four Sector Economy," Measures To Correct Market Failure.

[19] During the Great Recession, the government expenditure multiplier was high, meaning government investment was the most viable way to stimulate the economy. This idea came to fruition in the American Recovery and Reinvestment Act of 2009.

I didn't realize *just how much* they were. Obviously, this system is extremely limited. Going over the finer points of those limitations would be a paper unto itself. This model won't shed light on outside sources that can cause a change within the system. This is meant to be a very general approach to large macroeconomic trends. I look forward to future classes with Mr. Dissanayake, and more in-depth excursions into these fundamental ideas.

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